Commutativity Reasoning for the Heap

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Commutativity in Context

Used in increasingly many tools and theories!

- Scalable concurrency
- Smart contracts
- Transactional memory
- Automatic parallelization

… What’s commutativity?

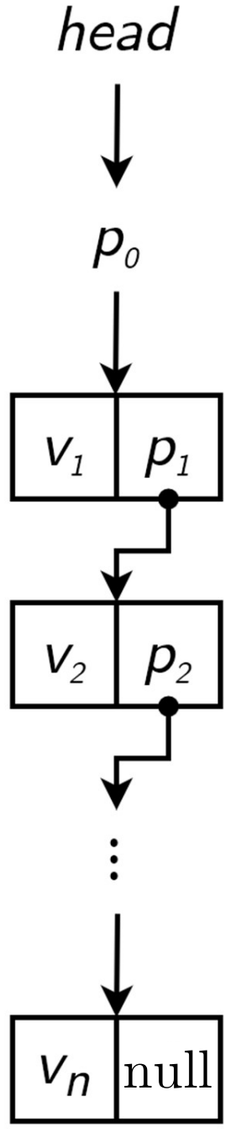
Motivating Example: Linked Stack

Consider a **linked stack** with `push`, `pop`, and `peek` operations.

```
push(v):
  h = ! head
  p = (v, h)
  r = ref p
  h <- r
```

```
pop():
  h = ! head
  if h <> null
  then
    (v, p) = ! h
    head <- p
    free h
```

```
peek():
  h = ! head
  (v, _) = ! h
  return v
```



Given two operations on the stack, when can we **swap** their **order of execution** and maintain the **same outcome**?

Standard Definition of Commutativity

Define **programs** to have type: $\text{input vector} \to \text{state} \to \text{state}$.

Assume **determinism** and **termination**, so programs are well-defined.

Consider a predicate P on a state and two vectors of values,
and an **observational equivalence** relation $\sim$.

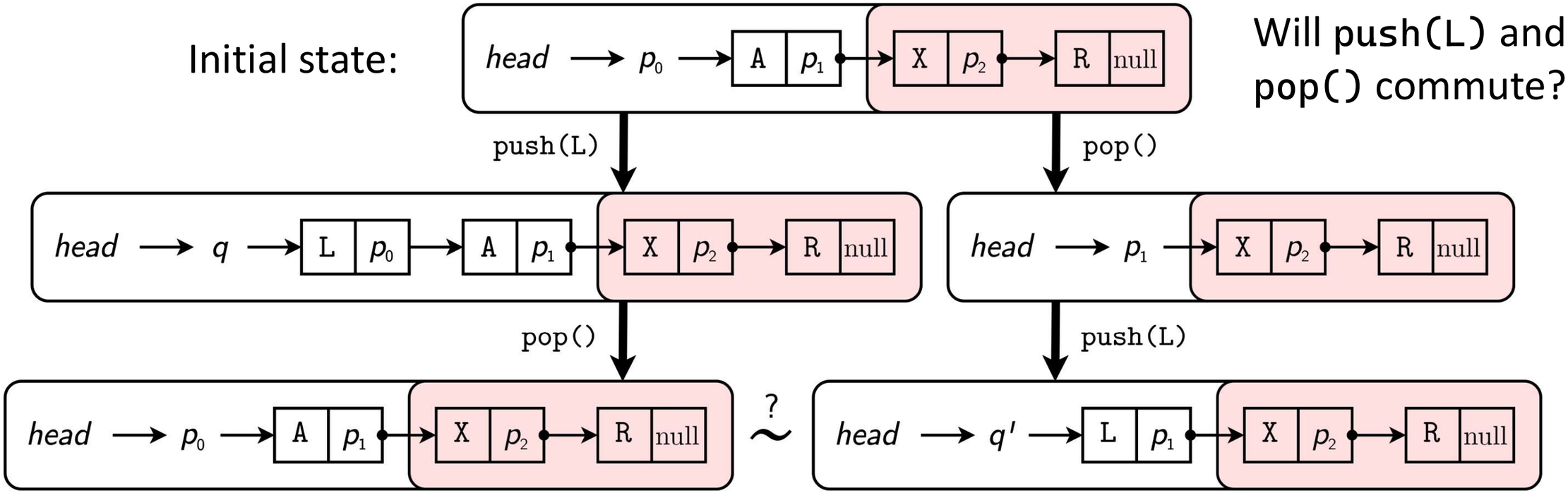
Equivalence modulo *observation*, not exact equality

Two programs C_1 and C_2 **commute** under P, denoted $C_1 \bowtie_P C_2$, if

$$\forall \sigma\ \bar{x}\ \bar{y}\ .\ P\ \sigma\ \bar{x}\ \bar{y}\ \Longrightarrow\ C_2\ \bar{y}\ (C_1\ \bar{x}\ \sigma) \sim C_1\ \bar{x}\ (C_2\ \bar{y}\ \sigma).$$

When $C_1 \bowtie_P C_2$ holds, we call P a correct **commutativity condition** for C_1 and C_2.

Motivating Example: Linked Stack



The bottom cells of the stack remain unread and unwritten.
But we still check both entire states for equivalence!

How can we **frame away** untouched pieces of state?

Key Contributions

Commutativity with Heaps

Separation Logic

Abstraction Predicates

Framed Commutativity

Implementation in Coq

Software Foundations Volume 6

Automation Tactics

Correctness and Completeness Proofs

Veracity Interpreter

Commute Blocks

Runtime with Parallelization

Heap Integration

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Separation Logic: Heap Predicates

Separation logic concerns **pieces of state** called **heaps**:
finite maps from **addresses** to **values**.

Heaps are characterized using **heap predicates**.

$$\frac{}{\mathbf{emp}\ \emptyset}\ \text{Empty} \qquad \frac{}{(p \mapsto v)\ (p{:}\, v)}\ \text{Single} \qquad \frac{P}{\forall h\ .\ [P]\ h}\ \text{Pure}$$

$$\frac{P\ h_1 \quad Q\ h_2 \quad h_1 \cap h_2 = \emptyset}{(P * Q)\ (h_1 \cdot h_2)}\ \text{Separating Conjunction}$$

Separation Logic: Heap Triples

Heap triples are the Hoare triples of separation logic.

Given program C and heap predicates P and Q, $\{P\}\ C\ \{Q\}$ holds if $\forall h\,.\,P\ h \Rightarrow Q\ (C\ h)$.

$$\frac{P' \Rightarrow P \quad \{P\}\ C\ \{Q\} \quad Q \Rightarrow Q'}{\{P'\}\ C\ \{Q'\}}$$ Rule of Consequence

$$\frac{\{P\}\ C\ \{Q\}}{\{P * R\}\ C\ \{Q * R\}}$$ Frame Rule

Ex: $\{x \mapsto 5\}\ x := 2\ \{x \mapsto 2\}$

$\{x \mapsto 5 * y \mapsto 7\}\ x := 2\ \{x \mapsto 2 * y \mapsto 7\}$

$\{\mathbf{emp}\}\ \text{ref}\ 5\ \{\exists p\,.\,p \mapsto 5\}$

Framed Commutativity: Attempt 1

$$\{P\ \bar{x}\ \bar{y} * \mathbf{H}\}$$

$$C_1\ \bar{x}; C_2\ \bar{y} \ \Big|\ C_2\ \bar{y}; C_1\ \bar{x}$$

$$\{Q * \mathbf{H}\}$$

P is the **commutativity condition** on the **program inputs** and **initial heap**.

$\mathbf{H}$ is easily framed away.

Both post-heaps must satisfy Q ~~– equivalence!~~

This doesn't work! Q isn't precise enough.

If Q **lacks existentials**, h_1 and h_2 need to be **identical**.

If Q **includes existentials**, the **relationship** between h_1 and h_2 is severed.

Framed Commutativity: Attempt 2

$\{P\ \bar{x}\ \bar{y} * \mathbf{H}\}$	$\{P\ \bar{x}\ \bar{y} * \mathbf{H}\}$
$C_1\ \bar{x}; C_2\ \bar{y}$	$C_2\ \bar{y}; C_1\ \bar{x}$
$\{Q_1 * \mathbf{H}\}$	$\{Q_2 * \mathbf{H}\}$

$\sim$

Now h_1 and h_2 satisfy distinct postconditions.

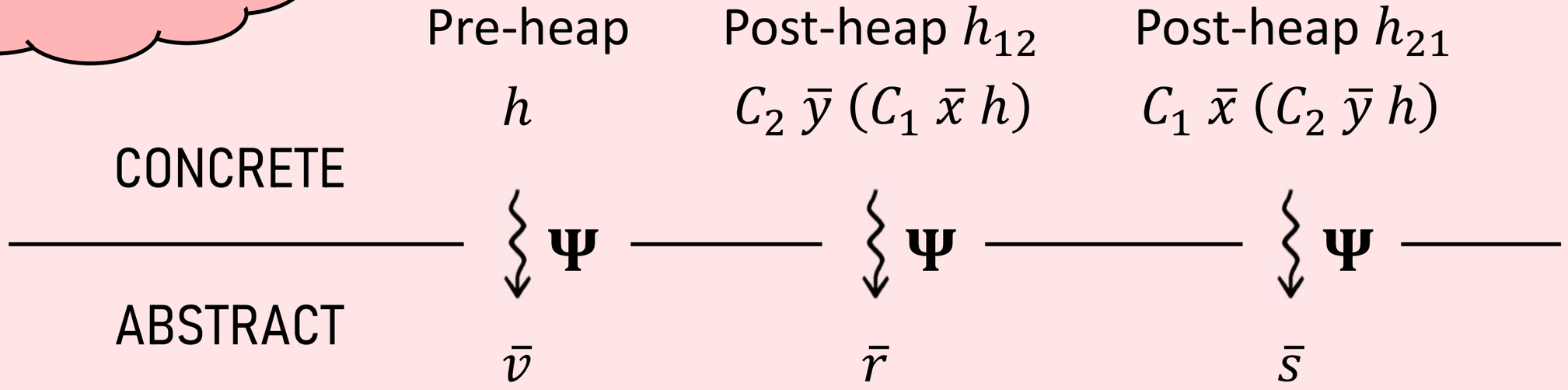
But we still haven't related h_1 and h_2!

Derive relatable values from postconditions?

Bottom line: heap triples aren't cutting it…

Abstraction Predicates

$P\ \bar{v} \Longrightarrow \bar{r} \sim \bar{s}$



$\mathbf{\Psi}$ is a problem-specific **abstraction predicate** parameterized on an **abstraction vector**.

Ex: Non-negative counter $\quad \mathbf{\Psi}\ \langle v \rangle \coloneqq c \mapsto v * [v \geq 0]$

Two-set $\quad \mathbf{\Psi}\ \langle u, v \rangle \coloneqq p \mapsto u * q \mapsto v * [u \neq v]$

Framed Commutativity

$$\begin{array}{lll} & \boldsymbol{\Psi} & \\ h & \rightsquigarrow & \bar{v} \\ h_{12} & \rightsquigarrow & \bar{r} \\ h_{21} & \rightsquigarrow & \bar{s} \end{array}$$

$$h_{12} \coloneqq C_2\ \bar{y}\ (C_1\ \bar{x}\ h)$$
$$h_{21} \coloneqq C_1\ \bar{x}\ (C_2\ \bar{y}\ h)$$

$P\ \bar{v} \Longrightarrow \bar{r} \sim \bar{s}$

Ingredients:

C_1, C_2	Programs
$\boldsymbol{\Psi}$	Abstraction predicate
$\sim$	Eq. relation on abstraction vectors
$\mathcal{P}$	Predicate on 3 vectors

C_1 and C_2 **commute** under $\mathcal{P}$, denoted $C_1 \bowtie_{\mathcal{P}} C_2$, if

$$\forall \mathbf{H}\ \bar{v}\ \bar{x}\ \bar{y}\ h\ .\ \underbrace{(\boldsymbol{\Psi}\ \bar{v} * \mathbf{H})\ h\ \wedge\ \mathcal{P}\ \bar{v}\ \bar{x}\ \bar{y}}_{\text{PRECONDITION}} \Longrightarrow \underbrace{\exists \bar{r}\ \bar{s}\ .\ (\boldsymbol{\Psi}\ \bar{r} * \mathbf{H})\ h_{12} \wedge (\boldsymbol{\Psi}\ \bar{s} * \mathbf{H})\ h_{21} \wedge \bar{r} \sim \bar{s}}_{\text{POST-RELATION}}$$

We say that $\mathcal{P}$ is a **correct commutativity condition** for C_1 and C_2.

Motivating Example: Linked Stack

Consider again our linked stack with **push**, **pop**, and **peek** operations.

$\boldsymbol{\Psi}$ should encode a **partial stack** to some depth,
by observing the stack contents V and last pointer f before framed portion.

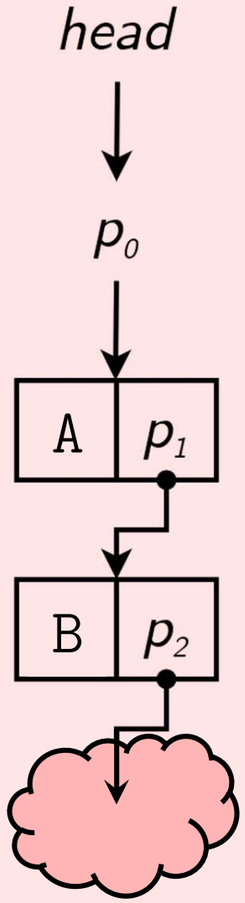
Abstraction for partial stack (top n values):

$$\boldsymbol{\Psi}_n\ \langle f, V\rangle \quad \coloneqq [|V| = n]\ *\ \exists p\ .\ head \mapsto p\ *\ \psi\ f\ p\ V$$
$$\psi\ f\ p\ (v :: t) \ \coloneqq\ \exists q\ .\ p \mapsto (v, q)\ *\ \psi\ f\ q\ t$$
$$\psi\ f\ p\ \mathrm{nil} \quad \coloneqq\ [p = f]$$

Abstraction for total stack (all values):

$$\boldsymbol{\Psi}_\star\ V \coloneqq \boldsymbol{\Psi}_{|V|}\ \langle \mathrm{null}, V\rangle$$

$V = [\mathtt{A}, \mathtt{B}]$
$f = p_2$



Motivating Example: Linked Stack

For each pair of stack operations, select particular depths for Ψ_n in the precondition and post-relation.

Operation pair	Pre depth	Post depth	V	$\mathcal{P}$
`push(x)` ⋈ `push(y)`	0	2	`[]`	`x = y`
`push(x)` ⋈ `peek`	1	2	`[v]`	`v = x`
`push(x)` ⋈ `pop`	1	1	`[v]`	`v = x`
`peek` ⋈ `peek`	1	1	`[_]`	`T`
`peek` ⋈ `pop`	2	1	`[v1,v2]`	`v1 = v2`
`pop` ⋈ `pop`	2	0	`[_,_]`	`T`

We will only ever need a depth of 2 for all stack reasoning!

Complete Conditions

Correct condition formula:

$$\forall \mathbf{H}\ \bar{v}\ \bar{x}\ \bar{y}\ h\,.\ (\mathbf{\Psi}\ \bar{v} * \mathbf{H})\ h\ \wedge\ \mathcal{P}\ \bar{v}\ \bar{x}\ \bar{y} \implies \exists \bar{r}\ \bar{s}\,.\,(\mathbf{\Psi}\ \bar{r} * \mathbf{H})\ h_{12} \wedge (\mathbf{\Psi}\ \bar{s} * \mathbf{H})\ h_{21} \wedge \bar{r} \sim \bar{s}$$

Conversely, $\mathcal{P}$ is **complete** if

$$\forall \mathbf{H}\ \bar{v}\ \bar{x}\ \bar{y}\ h\,.\ (\mathbf{\Psi}\ \bar{v} * \mathbf{H})\ h \implies \exists \bar{r}\ \bar{s}\,.\,(\mathbf{\Psi}\ \bar{r} * \mathbf{H})\ h_{12} \wedge (\mathbf{\Psi}\ \bar{s} * \mathbf{H})\ h_{21} \wedge \bar{r} \sim \bar{s} \implies \mathcal{P}\ \bar{v}\ \bar{x}\ \bar{y}$$

Thus, $\mathcal{P}$ is **exact** (correct and complete) if

$$\forall \mathbf{H}\ \bar{v}\ \bar{x}\ \bar{y}\ h\,.\ (\mathbf{\Psi}\ \bar{v} * \mathbf{H})\ h \implies \left(\exists \bar{r}\ \bar{s}\,.\,(\mathbf{\Psi}\ \bar{r} * \mathbf{H})\ h_{12} \wedge (\mathbf{\Psi}\ \bar{s} * \mathbf{H})\ h_{21} \wedge \bar{r} \sim \bar{s} \iff \mathcal{P}\ \bar{v}\ \bar{x}\ \bar{y} \right)$$

Evaluation

As framed commutativity is partly **relational**, it does not play nicely with standard heap triples out of the box.

Framed commutativity is not easily composable,
e.g. deriving $a \bowtie b; c$ from $a \bowtie b$ and $a \bowtie c$ is not straightforward.

Relational separation logic may offer some solutions.

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Infrastructure

Coq implementation is built atop the separation logic library in Volume 6 of *Software Foundations*.

Library features a toy language with big-step semantics.

```
Example push v :=
  <{ let 'h = ! head in
     let 'p = `pair 'h v in
     let 'r = ref 'p in
     head := 'r;
     () }>.
```

```
Example pop :=
  <{ let 'h = ! head in
     let 'g = 'h = null in
     if 'g then () else
     let 'p = ! 'h in
     let 'q = fst 'p in
     head := 'q;
     free 'h;
     () }>.
```

```
Example peek (r : loc) :=
  <{ let 'h = ! head in
     let 'p = ! 'h in
     let 'v = snd 'p in
     r := 'v; () }>.
```

Developed Utilities

Objective: automate commutativity proofs as much as possible.

We developed a suite of tactics and algorithms, all mostly automated:

- Dispatch heap-based goals
- Fact derivation from heap-based hypotheses
- Forward big-step symbolic execution of program goals
- Inverted big-step symbolic execution of program hypotheses

Correctness Proof Structure

```
Example comm_c1_c2 : forall v x y,
  P v x y ->
  comm (c1 x) (c2 y) (psi v) R.
Proof.
```

Setup ⟶ `introv HP phi. unfolds R, psi, c1, c2.`

Derive facts about h from $(\mathbf{\Psi}\ \bar{v} * \mathbf{H})\ h$ ⟶ `destruct_heap phi.`

`(* Other simplifications here *)`

Forward symbolic execution 4 times ⟶ `eexists*; splits; apply_evals.`

Instantiate existentials of post-relation ⟶ `eexists*. splits;`

Dispatch $(\mathbf{\Psi}\ \bar{r} * \mathbf{H})\ h_{12}$ and $(\mathbf{\Psi}\ \bar{s} * \mathbf{H})\ h_{21}$ ⟶ `solve_hprop.`

Problem-specific dispatch of $\bar{r} \sim \bar{s}$ ⟶ `(* Solve r ~ s here *)`

```
Qed.
```

Completeness Proof Structure

```
Example comm_c1_c2_inv: forall v x y,
  comm (c1 x) (c2 y) (psi v) R ->
  P v x y.
Proof.
```

Setup	⟶	`introv C.`
Construct a heap h that satisfies $\mathbf{\Psi}\ \bar{v}$	⟶	`assert ((psi v) h) as H. { solve_hprop. }`
Instantiate `comm` with h and $\mathbf{H} \coloneqq \mathbf{emp}$	⟶	`apply_comm C H. unfolds R, psi, c1, c2.`
Inverted symbolic execution 4 times	⟶	`invert_evals.`
Derive facts about h_{12} and h_{21}	⟶	`destruct_heap HR1, HR2.`
Problem-specific facts from $\bar{r} \sim \bar{s}$	⟶	`(* Derive facts from r ~ s here *)`
Problem-specific dispatch of $\mathcal{P}$	⟶	`(* Dispatch P here *)`

```
Qed.
```

Evaluation

Functionality:

- Largely automated verification of $\mathcal{P}$ **correctness** and **completeness**
- Interactive $\mathcal{P}$ **inference** in some scenarios

Useful contexts:

- Reason about programs with multiple data structures for free
- Simple reasoning about inductive structures

Challenges:

- Requires full big-step symbolic execution
- Not compositional – every commutativity proof stands in isolation

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Commute Blocks

Veracity is a C-like language with the novel construct of the **commute block**.

```
commute (guard) {
  { f1 }
  { f2 }
}
```

```
commute (c >= 0) {
  { c = c + 1; }
  { if (c > 0) c = c - 1; }
}
```

`commute` has two equivalent semantics:

1. Sequential (`guard` is `false`): `f1; f2`
2. Concurrent (`guard` is `true`): Run `f1` and `f2` in parallel (with Multicore OCaml)

Look out for our presentation at NJPLS!

Heap Integration

Toolchain uses abstraction refinement to infer and verify `commute` block guards.

Interpreter (OCaml) encodes state as `(string * (value ref)) list`.

- Similar in principle to separation logic heaps

Possible heap integrations:

1. Encode separation logic in infer/verify toolchain
2. Coq theories ⟼ VCY interpreter
3. VCY programs ⟼ Coq AST

Summary

Framed Commutativity

- Novel abstraction for heap-based commutativity reasoning
- Simplified modeling/reasoning via framing

Coq Implementation

- Automation of framed commutativity proofs
- Room to develop more capabilities

Veracity Interpreter

- Commute block semantics
- Opportunities for heap integration

Thank you!

All comments and questions are welcome.

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Related Work

Papers that verify/discover/prove commutativity:

- Bansal, Koskinen, Tripp. “Automatic Generation of Precise and Useful Commutativity conditions.” TACAS’18
- Kim, Rinard. “Verification of Semantic Commutativity Conditions and Inverse Operations on Linked Data Structures.” PLDI’11
- Pîrlea, Kumar, Sergey. “Practical Smart Contract Sharding with Ownership and Commutativity Analysis.” PLDI’21
- Chen, Fathololumi, Koskinen, Pincus. “Veracity: Declarative Multicore Programming with Commutativity.” *Under review*

Papers that use commutativity:

- Diniz, Rinard. “Dynamic Feedback: An Effective Technique for Adaptive Computing.” PLDI’97
- Herlihy, Koskinen. “Transactional boosting: a methodology for highly-concurrent transactional objects.” PPoPP’08
- Clements et al. “The Scalable Commutativity Rule: Designing Scalable Software for Multicore Processors.” SOSP’13

Return Value Lifting

How do we reason about functions with return values?

Augment $\mathbf{\Psi}$ with addresses to which programs a and b write any returns.

$$\mathbf{\Psi}'\ \langle v_a, v_b, \bar{v}\rangle \ :=\ \rho_a \mapsto v_a * \rho_b \mapsto v_b * \mathbf{\Psi}\ \bar{v}$$

$$\langle r_a, r_b, \bar{r}\rangle \sim'\ \langle s_a, s_b, \bar{s}\rangle \ :=\ r_a = s_a \wedge r_b = s_b \wedge \bar{r} \sim \bar{s}$$

Path Sensitivity

Recall the example of a two-element set, that stores non-negative integers at p and q.

```
Example mem (r : loc) (v : int) :=
  <{ let 'vp = ! p in
     let 'vq = ! q in
     let 'ep = ('vp = v) in
     let 'eq = ('vq = v) in
     if 'ep then
       (r := true; ())
     else if 'eq then
       (r := true; ())
     else
       (r := false; ()) }>.
```

```
Definition psi (u v : int) (t1 t2 : bool) :=
  p ~~> u \* q ~~> v \*
  \[u >= 0 \/ v >= 0 -> u <> v] \*
  r1 ~~> t1 \* r2 ~~> t2.

Definition R H h1 h2 :=
  exists u1 v1 u2 v2 t1 t2,
  (psi u1 v1 t1 t2 \* H) h1 /\
  (psi u2 v2 t1 t2 \* H) h2 /\
  ((u1 = u2 /\ v1 = v2) \/ (u1 = v2 /\ v1 = v2)).
```

To prove `mem` $\bowtie_{\top}$ `mem`, we must account for ITE branching!

Preemptive Guard Destruction

```
Example mem (r : loc) (v : int) :=
  <{ let 'vp = ! p in
     let 'vq = ! q in
     let 'ep = ('vp = v) in
     let 'eq = ('vq = v) in
     if 'ep then
       (r := true; ())
     else if 'eq then
       (r := true; ())
     else (r := false; ()) }>.
```

Naïve solution:

Destruct on every guard *before* existential instantiation

Kill any paths which contradict $\mathcal{P}$ hypothesis

Perform up to $2^{\#\,\text{guards}}$ proofs in parallel

```
Example comm_mem_mem :
  forall (u v x y : int) (t1 t2 : bool),
  x >= 0  ->  y >= 0 ->
  comm (mem r1 x) (mem r2 y)
       (psi u v t1 t2) R.
Proof.

  introv Zx Zy S. unfolds mem, R, psi.
  destruct_heap S. simpl_disjoint.

  either (u = x); either (v = x);
  either (u = y); either (v = y); subst.

  all: try contradiction.
  all: try (exfalso; apply~ L; fail).

  all: solve_comm.
  all: eexists*; splits.
  all: solve_hprop.
  all: auto.

Qed.
```